Trigonometric Integrals

Powers of Sine & Cosine (Wallis's Formulas)

- If n is odd and n ≥ 3

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \int_0^{\frac{\pi}{2}} \cos^n x \, dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right)$$

- If n is even (n ≥ 2)

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \int_0^{\frac{\pi}{2}} \cos^n x \, dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$$

Combined Sine & Cosine

Following are techniques for evaluating integrals of the form

$$\int \sin^m x \cos^n x \, dx$$

where *m* and *n* are positive integers.

Three Cases

- Power of sine is odd
  - Save one sine factor for the *du*, convert the remaining factors to cosines, then expand and integrate.

$$\int \sin^{2k+1} x \cos^n x \, dx \;\rightarrow\; \int (\sin^2 x)^k \cos^n x \sin x \, dx \;\rightarrow\; \int (1-\cos^2 x)^k \cos^n x \sin x \, dx$$

Do *u*-substitution with $u = \cos x$

- Power of cosine is odd
  - Save one cosine factor for the *du*, convert the remaining factors to sines, then expand and integrate.

$$\int \sin^m x \cos^{2k+1} x \, dx \;\rightarrow\; \int \sin^m x (\cos^2 x)^k \cos x \, dx \;\rightarrow\; \int \sin^m x (1-\sin^2 x)^k \cos x \, dx$$

Do *u*-substitution with $u = \sin x$

- Powers of sine and cosine are both even
  - Repeatedly use the identities

$$\sin^2 x = \frac{1-\cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

until you get an odd power of cosine, then apply the second technique, above.

Sine-Cosine Products with Different Angles

Use the product-to-sum identities:

- $\sin(mx)\sin(nx) = \frac{1}{2}(\cos[(m-n)x] - \cos[(m+n)x]$
- $\cos(mx)\cos(nx) = \frac{1}{2}(\cos[(m-n)x] + \cos[(m+n)x])$
- $\sin(mx)\cos(nx) = \frac{1}{2}(\sin[(m-n)x] + \sin[(m+n)x])$

Tangent & Secant

Following are techniques for evaluating integrals of the form

$$\int \sec^m x \tan^n x \, dx$$

where m and n are positive integers.

Useful Standard Integrals

You'll need these:

$$\int \tan x \, dx = \ln|\sec x| + c$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + c$$

$$\int \sec^2 x \, dx = \tan x + c$$

Five Cases

- **Power of secant is even and ≥ 4** (tangent may or may not be present)
  - Save one $\sec^2$ factor for the du, convert the remaining factors to tangents, then expand, distribute the $\tan^n x$ (if any), and integrate.

$$\int \sec^{2k} x \tan^n x \, dx \rightarrow \int (\sec^2 x)^{k-1} \tan^n x \sec^2 x \, dx \rightarrow \int (1 + \tan^2 x)^{k-1} \tan^n x \sec^2 x \, dx$$

Expand this — Distribute this — Then do u-sub with u = tan x

- **Power of secant is odd** (no tangent factor)
  - Separate out a secant-squared factor and apply integration by parts

$$\int \sec^n x \, dx \rightarrow \int \sec^{n-2} x \sec^2 x \, dx \rightarrow \text{Let } u = \sec^{n-2} x \text{ and } dv = \sec^2 x \, dx$$

- **Power of tangent is even and ≥ 4** (no secant factor)
  - Convert a $\tan^2$ factor to a $\sec^2$ factor, then expand, distribute, and integrate.

$$\int \tan^n x \, dx \rightarrow \int \tan^{n-2} x (\tan^2 x) \, dx \rightarrow \int \tan^{n-2} x (\sec^2 x - 1) \, dx$$

You may need to apply this rule again once you distribute the tan.

- **Power of tangent is odd** (secant is present)
  - Pull out a secant-tangent factor and convert the remaining factors to secants, then expand, distribute, and integrate.

$$\int \sec^m x \tan^{2k+1} x \, dx \rightarrow \int (\sec^{m-1} x)(\tan^2 x)^k \tan x \sec x \, dx \rightarrow \int (\sec^{m-1} x)(\sec^2 x - 1)^k \tan x \sec x \, dx$$

Do u-sub with u = sec x

- **Power of tangent is odd** (no secant factor)
  - Pull out a tangent, convert the other tangents to secants, then expand, distribute, and integrate

$$\int \tan^{2k+1} x \, dx \rightarrow \int (\tan^2 x)^k \tan x \, dx \rightarrow \int (\sec^2 x - 1)^k \tan x \, dx$$

Distribute the tan, the apply other rules, as needed.

- **None of the above**
  - Convert everything to sine and cosine; no guarantees, but it's likely to give you something useful.