Vectors

Vector Notation

In printed material, vector names are indicated by heavy text: **A**.
In handwriting, vectors are usually indicated by an arrow above the name: $\vec{A}$

$\mathbf{A} = \langle x, y \rangle$	Vector with specified *x*- and *y*-components
$\mathbf{B} = x\mathbf{i} + y\mathbf{j}$	Vector with specified unit-vector components
$\lVert\mathbf{A}\rVert$ or $[\![\mathbf{A}]\!]$	Magnitude (*i.e.*, length) of vector **A**.

Basic Operations

In the following, $\mathbf{A} = \langle a_1, a_2 \rangle$, $\mathbf{B} = \langle b_1, b_2 \rangle$

Addition, subtraction, scalar multiplication

$\mathbf{A} + \mathbf{B} = \langle a_1 + b_1, a_2 + b_2 \rangle$

$\mathbf{A} - \mathbf{B} = \langle a_1 - b_1, a_2 - b_2 \rangle$

$c\mathbf{A} = \langle ca_1, ca_2 \rangle$ ← *c* is a scalar constant (a.k.a., a number).

Dot Product

The dot product of two vectors is a scalar value, *i.e.*, a number, not a vector.

- $\mathbf{A} \bullet \mathbf{B} = a_1 b_1 + a_2 b_2$
- $\mathbf{A} \bullet \mathbf{B} = \lVert\mathbf{A}\rVert \, \lVert\mathbf{B}\rVert \cos(\theta)$ ← θ is the angle between the two vectors

  Note that $\mathbf{A} \bullet \mathbf{A} = \lVert\mathbf{A}\rVert^2$

Dot Product and Angle

The sign of $\mathbf{A} \bullet \mathbf{B}$ indicates the nature of the angle between those vectors:

$\mathbf{A} \bullet \mathbf{B} < 0$	Acute
$\mathbf{A} \bullet \mathbf{B} = 0$	Right
$\mathbf{A} \bullet \mathbf{B} > 0$	Obtuse

Other Vector Calculations

Vector between two points

Given points $A(x_1, y_1)$ and $B(x_2, y_2)$,

- Vector $\mathbf{AB} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j}$

Unit Vector

$\hat{\mathbf{a}}$, the unit vector parallel to $\mathbf{A} = \langle x, y \rangle$, is

- $\hat{\mathbf{a}} = \dfrac{x}{\lVert\mathbf{A}\rVert}\mathbf{i} + \dfrac{y}{\lVert\mathbf{A}\rVert}\mathbf{j}$

Perpendicular vs Orthogonal vs Normal

These three words mean essentially the same thing: intersecting at a right angle. Commonly, however, they are used to apply to different circumstances:

Orthogonal	Two vectors
Perpendicular	Two lines or two planes
Normal	A vector and a line or plane

Projection of a onto b

This is the component of **a** in the direction of **b**

$$\text{proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{b}||^2} \times \mathbf{b}$$

Scalar projection of a onto b

$$\frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{b}||} = ||\mathbf{a}|| \cos(\theta)$$

Cross Product

The cross product results in a new vector that is perpendicular to the original vectors. Note that the cross product is applicable only to three-dimensional vectors.

- *Magnitude:* $\mathbf{A} \times \mathbf{B} = ||\mathbf{A}||\ ||\mathbf{B}|| \sin(\theta)$
- *Direction:* Orthogonal to both the original vectors using the right-hand rule for $\mathbf{A} \rightarrow \mathbf{B}$

Parallel Vectors

If **A** and **B** are parallel or antiparallel, then

$$\mathbf{A} \times \mathbf{B} = 0$$

Formal Definition of cross product

- $$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Algebraic Properties of the Cross Product

- $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
- $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
- $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$
- $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
- $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
- $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

Geometric Properties of the Cross Product

- $\mathbf{u} \times \mathbf{v}$ is orthogonal to both **u** and **v**
- $||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}||\ ||\mathbf{v}|| \sin\theta$
- $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ iff **u** is a scalar multiple of **v**
- $||\mathbf{u} \times \mathbf{v}||$ = area of a parallelogram having **u** and **v** as adjacent sides

Triple Scalar Product

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Geometry & Triple Scalar Product

The volume, V, of a parallelepiped with **u**, **v**, and **w** as adjacent edges is

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$$